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# On the Linearization of Nonlinear Langevin-Type Stochastic Differential Equations

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We show that a logical extension of the piecewise optimal linearization procedure leads to the Gaussian decoupling scheme, where no iteration is required. The scheme is equivalent to solving a few coupled equations. The method is applied to models which represent (a) a single steady state, (b) passage from an initial unstable state to a final preferred stable state by virtue of a finite displacement from the unstable state, and (c) a bivariate case of passage from an unstable state to a final stable state. The results are shown to be in very good agreement with the Monte Carlo calculations carried out for these cases. The method should be of much value in multidimensional cases.

**KEY WORDS:** Nonlinear Langevin equation; statistical linearization; piecewise optimal linearization; Gaussian decoupling; Monte Carlo calculation; plastic flow; yield drop.

## 1. INTRODUCTION

Nonlinear stochastic differential equations arise in modeling of many physical processes.[1,2] Generally, these equations are not amenable to closed form solutions. Perforce one has to resort to approximate procedures. The only exception is the Monte Carlo technique, which is numerically exact. But the Monte Carlo method is not attractive since it is computationally expensive.

Many of the approximate procedures devised are based on the concept of linearization. One such method is the statistical (or equivalent) linearization procedure.[3-6] In this technique the nonlinear equation is replaced by

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a linear one in such a way that the ensemble average of the square of the error due to the replacement is minimum.[3] Recently Eaves and Reinhardt[7] suggested a modification wherein the linearization was carried out piecewise at different time intervals. The solution of the linearized equation in a given time interval was used to average the square of the error. These authors applied the piecewise optimal linearization (POL) procedure to a Langevin equation with one steady state which is stable and showed that its performance is much better than the statistical linearization procedure. None of these procedures are applicable to cases where the passage is from an unstable state to stable states.

There are a class of problems where the passage from an unstable state can be either restricted to a single steady state (by an appropriate choice of initial conditions) or the inherent nature of the problem allows only one steady state. In such situations both the first and second moments would be of interest. In the case when the driving noise is Gaussian, we show that a logical extension of the POL procedure gives a Gaussian decoupling scheme. This scheme is uniformly good for all times. The simplicity of the method lies in the few coupled equations one has to solve (two in the case of a single variable and five in the case of two variables). The method appears to be very useful whenever the interest is restricted to the first two moments. (In most cases of physical interest this should be adequate.) The method should be of much value in multidimensional cases when the Monte Carlo technique becomes prohibitive. The method also throws light on situations where there are multiple steady states.

We consider models which represent (a) a single steady state, (b) passage from an initial unstable state to a preferred stable state by virtue of a finite displacement from the unstable state, and (c) passage from an unstable state to a single stable state. The first two are single-variable cases and the last is a two-variable case. We also briefly examine the passage from an unstable state to multiple steady states studied by Suzuki.[8,9] For comparison of the accuracy of the suggested decoupling scheme, in each of the examples considered we have also performed the Monte Carlo calculation.

The plan of the paper is as follows: In Section 2, we outline the statistical linearization and the POL. We show how this procedure naturally leads to the Gaussian decoupling scheme for a driving Gaussian white noise. The example considered is the overdamped Bernoulli oscillator (allowing for both signs in the linear term). We consider both the stable and unstable cases. Comparison with the Monte Carlo results demonstrates that in both (a) and (b) cases, the time dependence of the first two moments is predicted very well. A Gaussian bimodal choice for the decoupling indi-

cates why Suzuki's self-consistent method works. In Section 3, we consider a two-variable Langevin equation which has relevance in a plastic flow situation.[10,11] A bivariate Gaussian distribution is used to decouple the system of equations. Again it is demonstrated that the results of the first two moments agree very well with the Monte Carlo results. Section 4 is devoted to discussion. For the bivariate Langevin equation we have shown the equivalence of the POL procedure and the Gaussian decoupling scheme. This has been outlined in the Appendix.

## 2. STATISTICAL LINEARIZATION AND PIECEWISE OPTIMAL LINEARIZATION

In this section we discuss a model described by the equation

$$\frac{dx}{dt} + \beta x + gx^3 = \eta(t) \tag{1}$$

which represents an overdamped Bernoulli oscillator. $\eta(t)$ is a Gaussian white noise with

$$\langle \eta(t) \rangle = 0 \quad \text{and} \quad \langle \eta(t)\eta(t') \rangle = 2\epsilon\delta(t - t') \tag{2}$$

where $2\epsilon$ is the strength of the random force.[3] $g$ is always positive to ensure global stability. If $\beta > 0$, $x = 0$ is the only steady state which is stable. If $\beta < 0$, there are three steady states $x = 0, \pm(\gamma/g)^{1/2}$ where $\gamma = -\beta$. The steady state $x = 0$ is unstable. In the following we investigate the case $\beta > 0$ and also the case $\beta < 0$ but with the initial point displaced by a value much larger than $\epsilon^{1/2}$ so that the system preferentially evolves to one of the stable steady states. There are many physical situations where a finite displacement is of interest due to the inherent nature of the problem. (For example the initial value of the dislocation density in a plastic flow problem is finite. See for instance Ref. 10.)

The statistical linearization procedure[3-6] consists in approximating the nonlinear Eq. (1) to the linear one given by

$$\frac{dx(t)}{dt} + \tilde{\beta}x(t) = \eta(t) \tag{3}$$

The error committed due to this replacement is

$$\Delta(x) = (\beta - \tilde{\beta})x + gx^3 \tag{4}$$

$\tilde{\beta}$ in Eqs. (3) and (4) is determined by demanding that the ensemble

[3] In physical examples, the value of $\epsilon$ is of the order of the inverse of the size of the system. See Refs. 8 and 9.

average of the square of the error be minimum, i.e.,

$$\frac{\partial}{\partial \tilde{\beta}} \langle \Delta^2(x) \rangle = 0 \tag{5}$$

The bracket $\langle\ \rangle$ in Eq. (5) represents the average over the equilibrium distribution under the assumption that $\Delta^2$ is ergodic. When the equilibrium distribution is not known, the average in Eq. (5) is replaced[4] by an average over the approximate equilibrium distribution obtained as a solution of the linearized Eq. (3).

Recently Eaves and Reinhardt[7] suggested a modification of the above linearization. The time is partitioned into a set of intervals each of arbitrary length. In the $j$th interval Eq. (1) is approximated as

$$\frac{dx(t)}{dt} + \tilde{\beta}_j x(t) + C_j = \eta(t), \qquad t_{j-1} < t < t_j \tag{6}$$

The error due to replacement of Eq. (1) by Eq. (6) is

$$\Delta_j(x) = (\beta - \tilde{\beta}_j)x - C_j + gx^3 \tag{7}$$

The parameters $\tilde{\beta}_j$ and $C_j$ are determined by demanding that

$$\frac{\partial}{\partial \tilde{\beta}_j} \langle \Delta_j^2(x) \rangle_{t_j} = \frac{\partial}{\partial C_j} \langle \Delta_j^2(x) \rangle_{t_j} = 0 \tag{8}$$

The bracket $\langle\ \rangle_{t_j}$ represents the average taken using the probability density function at time $t = t_j$. The expression for $\tilde{\beta}_j$ and $C_j$ are given by

$$\tilde{\beta}_j = \beta + g\left[\langle x^4 \rangle_{t_j} - \langle x^3 \rangle_{t_j} \langle x \rangle_{t_j}\right] / \sigma_j^2(t_j) \tag{9}$$

and

$$C_j = g\left[\langle x^2 \rangle_{t_j} \langle x^3 \rangle_{t_j} - \langle x \rangle_{t_j} \langle x^4 \rangle_{t_j}\right] / \sigma_j^2(t_j). \tag{10}$$

It may be pointed out that the above equations reproduce the exact rate equations for the first two moments obtained by starting from Eq. (1) with $\langle x(t)\eta(t) \rangle$ taken as $\epsilon$ to the leading order. Since Eqs. (9) and (10) contain up to four moments, there appears to be a need to calculate equations for moments up to the fourth one. However, $\eta(t)$ is Gaussian and therefore equations for the first two moments are sufficient. Propagation formulas for these can be easily obtained from Eq. (6). These authors impose the self-consistency condition to obtain $\tilde{\beta}_j$ and $C_j$ iteratively. However, since the solution of Eq. (6) is a Gaussian, $\tilde{\beta}_j$ and $C_j$ are given by

$$\tilde{\beta}_j = \beta + 3g\langle x^2 \rangle_{t_j} \tag{11}$$

and

$$C_j = -2g\langle x \rangle_{t_j}^3 \tag{12}$$

In deriving these equations, we have made use of the fact that

$$\langle x^3 \rangle_{t_j} = 3\langle x \rangle_{t_j} \langle x^2 \rangle_{t_j} - 2\langle x \rangle_{t_j}^3 \tag{13}$$

and

$$\langle x^4 \rangle_{t_j} = 3\langle x^2 \rangle_{t_j}^2 - 2\langle x \rangle_{t_j}^4 \tag{14}$$

It is obvious now that once we decide to use the solution of the linearized Eq. (6) for calculating $\tilde{\beta}_j$ and $C_j$, we can drop the subscript and regard $\tilde{\beta}$ and $C$ as continuous functions of time. (This is equivalent to $\Delta t_j = t_j - t_{j-1} \to 0$.) This permits a decoupling scheme where equations for the first two moments are sufficient. Using the expressions for $\tilde{\beta}$ and $C$ we get

$$\frac{dx(t)}{dt} + \left[ \beta + 3g\langle x^2(t) \rangle \right] x(t) - 2g\langle x(t) \rangle^3 = \eta(t) \tag{15}$$

where $\langle x^2(t) \rangle$ should be evaluated self consistently, i.e.,

$$\frac{d}{dt}\langle x(t) \rangle + \beta \langle x(t) \rangle + 3g\langle x(t) \rangle \langle x^2(t) \rangle - 2g\langle x(t) \rangle^3 = 0 \tag{16}$$

$$\frac{d}{dt}\langle x^2(t) \rangle + 2\beta \langle x^2(t) \rangle + 6g\langle x^2(t) \rangle^2 - 2g\langle x(t) \rangle^4 = 2\epsilon \tag{17}$$

In obtaining Eq. (17) we have made use of the fact that $\langle x(t)\eta(t) \rangle$ is $\epsilon$ [using the linear Equation (15)]. We could have obtained the above equations starting from the equations for the first two moments [obtained from Eq. (1)] and using the relations connecting the third and fourth moments to the first two. [Of course we have to use also the fact that $\langle x(t)\eta(t) \rangle$ is $\epsilon$ to the leading order.]

For the sake of computation we chose $\beta = g = 1$. It is clear that the system will relax asymptotically to the only steady state $x = 0$. We chose $\epsilon = 5 \times 10^{-7}$, $\langle x(0) \rangle = 2$ and $\langle x^2(0) \rangle = 4$ (i.e., deterministic initial conditions).

We have used fourth-order Runge–Kutta–Gill method to solve Eqs. (16) and (17). For the purpose of comparison we have also solved Eq. (1) using Monte Carlo method coupled with Runge–Kutta–Gill method. 4900 histories were generated and the convergence of the first two moments with increasing number of histories were explicitly tested. We have used POL procedure also as prescribed by Eaves and Reinhardt.[7] Needless to say, the POL procedure takes very much larger computer time by virtue of the iterations involved. Numerical solution of our equations for this problem takes approximately 3 sec for a relative accuracy of $1 \times 10^{-8}$. In contrast, the POL procedure takes approximately 10 min for a relative error of $1 \times 10^{-3}$ in $\tilde{\beta}_j$ and $C_j$ (with $\Delta t_j = 0.001$). Our results cannot be distinguished from the results of POL procedure. Apart from these three, we have also solved the problem with statistical linearization procedure

wherein we use the asymptotic solution of the linearized Eq. (3) to obtain self-consistently the time-independent coefficient $\tilde{\beta}$ (as suggested in Ref. 4). Figure 1 depicts the results on the first moment obtained by various methods. We see that the results of our decoupling scheme agree very well with the Monte Carlo results. The statistical linearization predicts a slower decay to the steady state compared to the exact Monte Carlo results. Figure 2 depicts the variance as a function of time. Again our results are in very good agreement with the Monte Carlo results. The results of the statistical linearization procedure deviate from the Monte Carlo results at the initial time considerably, but as $t \to \infty$ there is a better agreement as should be expected.

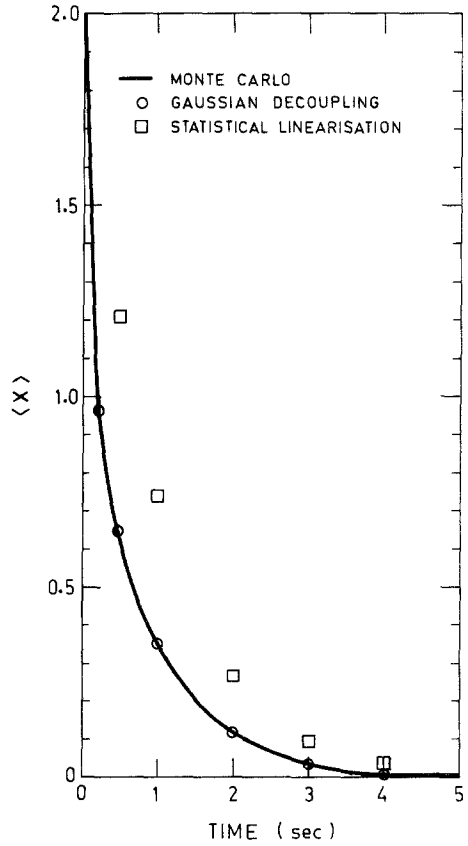


Fig. 1. $\langle x \rangle$ as a function of time for the stable case of the overdamped Bernoulli oscillator with $\epsilon = 5 \times 10^{-7}$ and deterministic initial condition $\langle x(0) \rangle = x(0) = 2$.

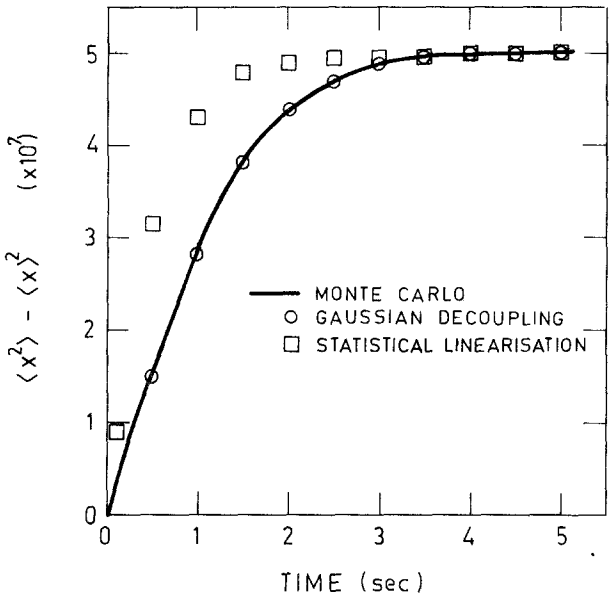


Fig. 2. The variance as a function of time for the process considered in Fig. 1.

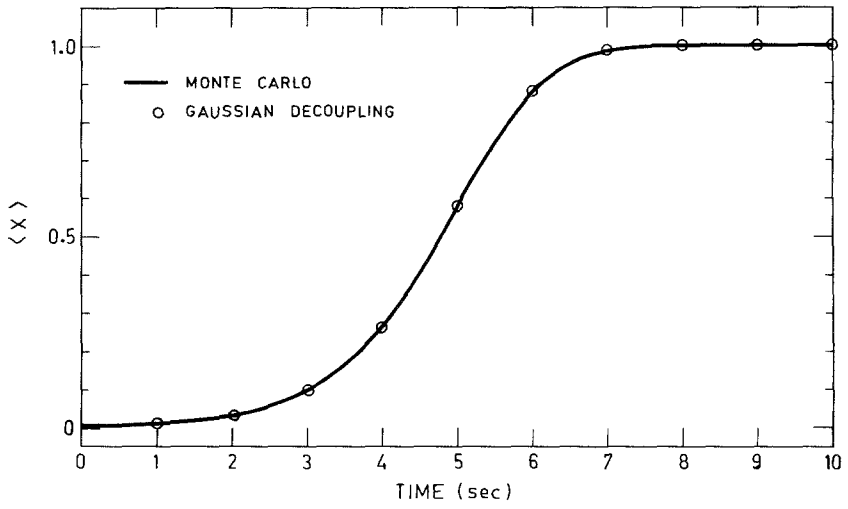


Fig. 3. $\langle x \rangle$ as a function of time for the unstable case which preferentially evolves to the positive stable steady state with the choice of parameters: $\epsilon = 5 \times 10^{-7}$, $\langle x(0) \rangle = 0.005$ and $\langle x^2(0) \rangle = 2.6 \times 10^{-5}$.

We now consider the unstable case $\beta = -\gamma$. This models a system that relaxes from the initial unstable state to a final stable state. We consider here two situations. First, we study the evolution from an initial state which is displaced from $x = 0$ by a value $\delta$ such that $\delta^2 \gg \epsilon$ so that the system evolves to the positive stable steady state preferentially. The situation corresponds to the extensive regime.[8,9] We have chosen $\langle x(0) \rangle = \delta = 5 \times 10^{-3}$ and $\epsilon = 5 \times 10^{-7}$. Figure 3 depicts the evolution of the mean as obtained by the Gaussian decoupling scheme and the Monte Carlo technique. Figure 4 depicts the evolution of the fluctuations. It is clear that the results of the decoupling scheme agree quite well except for a small discrepancy of the fluctuations in the intermediate region. This discrepancy

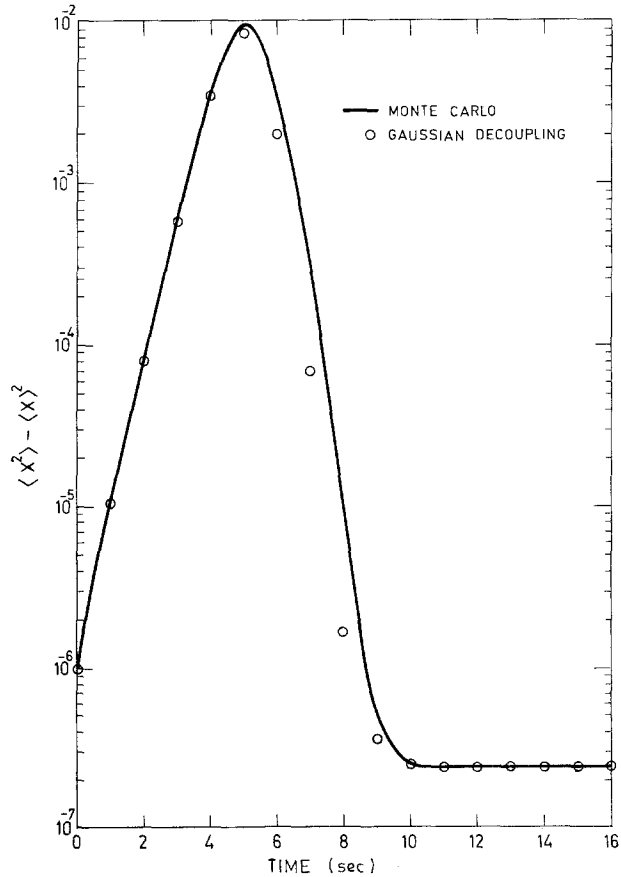


Fig. 4. The variance as a function of time for the situation considered in Fig. 3.

between the two is due to the non-Gaussian feature. The fluctuation enhancement characteristic of passage from an (apparent) unstable state to a stable state is predicted remarkably well by the Gaussian approximation. It should be pointed out that this is in a sense the generalization of Suzuki's decoupling scheme for $\langle x(0)\rangle \neq 0$. [Suzuki decouples $x^3 = \langle x^2\rangle x$. His decoupling scheme is applicable only when $\langle x(0)\rangle = 0$.] In the Monte Carlo calculations, 4900 tracks were generated and the convergence of the mean and the variance were explicitly verified. We have also carried out the POL procedure as prescribed by Eaves and Reinhardt[7] and the results match exactly for $\Delta t_j = 0.001$ with an error constraint on $\tilde{\beta}_j$ and $C_j$ equal to 0.1%.

Next we consider the situation $\langle x(0)\rangle = 0$. This corresponds to the relaxation from the intrinsic unstable region[8] ($\delta^2 \ll \epsilon$). In this case both the steady states are equally probable. The problem has been studied by Suzuki in detail using scaling theory.[8,9] As an illustration of the scaling property, he proposes a decoupling scheme which gives a second moment equation very different from the corresponding one in the extensive region. The results of this self-constraint decoupling scheme can be easily found to compare well with the Monte Carlo results.[12] Our interest here is to get some insight into why this decoupling scheme works. Here we will not be interested in comparing with the Suzuki scaling results. Our interest in the problem is to draw on the analysis of the problem and to suggest a decoupling procedure which will give reasonable results in the case when multiple steady states are allowed by any nonlinear Langevin equation.

It is easy to show that the Eqs. (16) and (17) will not hold in this case due to the strong non-Gaussian feature. From the studies of the earlier two situations, it is clear that the Gaussian decoupling scheme works quite well since the probability density has only one peak. In this problem, since there are two peaks which asymptotically go to $\pm(\gamma/g)^{1/2}$, a bimodal distribution would perhaps be a good choice. For the case of $\langle x(0)\rangle = 0$, since the distribution function is to be symmetric, we chose

$$P(x,t) = C\left\{H(-x)\exp\left[-\frac{(x+x_1)^2}{2\sigma_1^2}\right] + H(x)\exp\left[-\frac{(x-x_1)^2}{2\sigma_1^2}\right]\right\} \tag{18}$$

where

$$C = 1/(2\pi)^{1/2}\sigma_1\left\{1 + \operatorname{erf}\left[\frac{x_1}{(2\sigma_1)^{1/2}}\right]\right\}$$

Here $H(x)$ is the usual Heaviside step function, $x_1$ is the position of the peak, and $\sigma_1^2$ is the variance as defined for one part of the distribution.

Substituting for $\langle x^4 \rangle$ in

$$\frac{d}{dt}\langle x^2 \rangle = 2\left[\gamma\langle x^2 \rangle - g\langle x^4 \rangle + \epsilon\right] \tag{19}$$

we get

$$\frac{d}{dt}\langle x^2 \rangle = 2\Bigg\{\gamma\langle x^2 \rangle - g\langle x^2 \rangle^2 + \epsilon - 2g\sigma_1^2 \times\left[2\langle x^2 \rangle - \sigma_1^2 - Cx_1\langle x^2 \rangle \exp\left(-\frac{x_1^2}{2\sigma_1^2}\right)\right]\Bigg\} \tag{20}$$

Since $\sigma_1^2 = \langle x^2 \rangle_s - \langle x \rangle_s^2$ (where $s$ refers to one segment) we expect it to be small except perhaps in the intermediate regions. Our Monte Carlo results show that $\sigma_1^2$ is small compared to the other terms. Hence, to the first approximation we get Suzuki's decoupling scheme. This suggests that it is important to respect the symmetry of the distribution function.

## 3. THE BIVARIATE LANGEVIN EQUATION

The purpose of this section is to extend the results to nonlinear coupled Langevin equations. For the purpose of illustration we chose a model which has a physical basis in plastic flow.[10] Specifically, the model represents the yield drop phenomenon in materials like silicon. Elsewhere, we have used these coupled Langevin equations to calculate the physical properties during a yield drop.[11] The coupled set of equations is

$$\frac{dx}{dt} = xy - x^2 + \eta_1(t) \tag{21}$$

and

$$\frac{dy}{dt} = b_0 - b_1x^2y + b_2x^3 + \eta_2(t) \tag{22}$$

where $\eta_1$ and $\eta_2$ are taken to be Gaussian white noise with zero mean and

$$\langle \eta_i(t)\eta_j(t') \rangle = 2\epsilon_i\delta_{ij}\delta(t - t') \tag{23}$$

In the physical problem $x$ is a dimensionless variable related to the square root of the dislocation density and $y$ is related to the dimensionless stress. It is clear that there is only one stable steady state[4] given by $x = y = [b_0/(b_1 - b_2)]^{1/3}$. The constants $b_0$, $b_1$, and $b_2$ are material parameters.

[4] Generally, there is no steady state permitted in a constant strain rate experiment. Since our interest is to look at the fluctuations, we have constructed a model to permit a stable steady state.

It is straightforward to construct the equations of motion for the first two moments. These are given by

$$\frac{d}{dt}\langle x\rangle = \langle xy\rangle - \langle x^2\rangle \tag{24}$$

$$\frac{d}{dt}\langle y\rangle = b_0 - b_1\langle x^2y\rangle + b_2\langle x^3\rangle \tag{25}$$

$$\frac{d}{dt}\langle x^2\rangle = 2\left[\langle x^2y\rangle - \langle x^3\rangle + \epsilon_1\right] \tag{26}$$

$$\frac{d}{dt}\langle y^2\rangle = 2\left[b_0\langle y\rangle - b_1\langle x^2y^2\rangle + b_2\langle x^3y\rangle + \epsilon_2\right] \tag{27}$$

and

$$\frac{d}{dt}\langle xy\rangle = \langle xy^2\rangle - \langle x^2y\rangle + b_o\langle x\rangle - b_1\langle x^3y\rangle + b_2\langle x^4\rangle \tag{28}$$

In the above equations we have made use of the fact that

$$\langle x\eta_1\rangle = \epsilon_1, \qquad \langle y\eta_2\rangle = \epsilon_2 \tag{29}$$

$$\langle x\eta_2\rangle = 0 = \langle y\eta_1\rangle \tag{30}$$

which can be obtained in the linear approximation. To calculate these moments, we now express $\langle x^3\rangle$, $\langle x^4\rangle$, $\langle x^2y\rangle$, $\langle xy^2\rangle$, $\langle x^2y^2\rangle$ and $\langle x^3y\rangle$ in terms of the first two moments of each of the variables and the correlation coefficient using a bivariate Gaussian distribution. Since we are interested only in the first few moments, it is convenient to use the characteristic function,

$$\chi(\nu_1,\nu_2) = \exp\left[i\langle x\rangle\nu_1 + i\langle y\rangle\nu_2 - \tfrac{1}{2}\left\{\sigma_1^2\nu_1^2 + \sigma_2^2\nu_2^2 + 2\rho_{12}\sigma_1\sigma_2\nu_1\nu_2\right\}\right] \tag{31}$$

where

$$\sigma_1^2 = \langle x^2\rangle - \langle x\rangle^2 \tag{31a}$$

$$\sigma_2^2 = \langle y^2\rangle - \langle y\rangle^2 \tag{31b}$$

and

$$\rho_{12} = \frac{\langle xy\rangle - \langle x\rangle\langle y\rangle}{\sigma_1\sigma_2} \tag{31c}$$

Using

$$\langle x^n y^m\rangle = (-i)^{m+n}\frac{\partial^{n+m}}{\partial\nu_1^n\,\partial\nu_2^m}\left[\chi(\nu_1,\nu_2)\right]\Big|_{\nu_1=\nu_2=0} \tag{32}$$

we get

$$\langle x^3\rangle = 3\langle x\rangle\langle x^2\rangle - 2\langle x\rangle^3 \tag{33a}$$

$$\langle x^4\rangle = 3\langle x^2\rangle^2 - 2\langle x\rangle^4 \tag{33b}$$

$$\langle x^2y\rangle = 2\langle x\rangle[\langle xy\rangle - \langle x\rangle\langle y\rangle] + \langle x^2\rangle\langle y\rangle \tag{33c}$$

$$\langle xy^2\rangle = 2\langle y\rangle[\langle xy\rangle - \langle x\rangle\langle y\rangle] + \langle x\rangle\langle y^2\rangle \tag{33d}$$

$$\langle x^2y^2\rangle = \langle x^2\rangle\langle y^2\rangle + 2[\langle xy\rangle^2 - \langle x\rangle^2\langle y\rangle^2] \tag{33e}$$

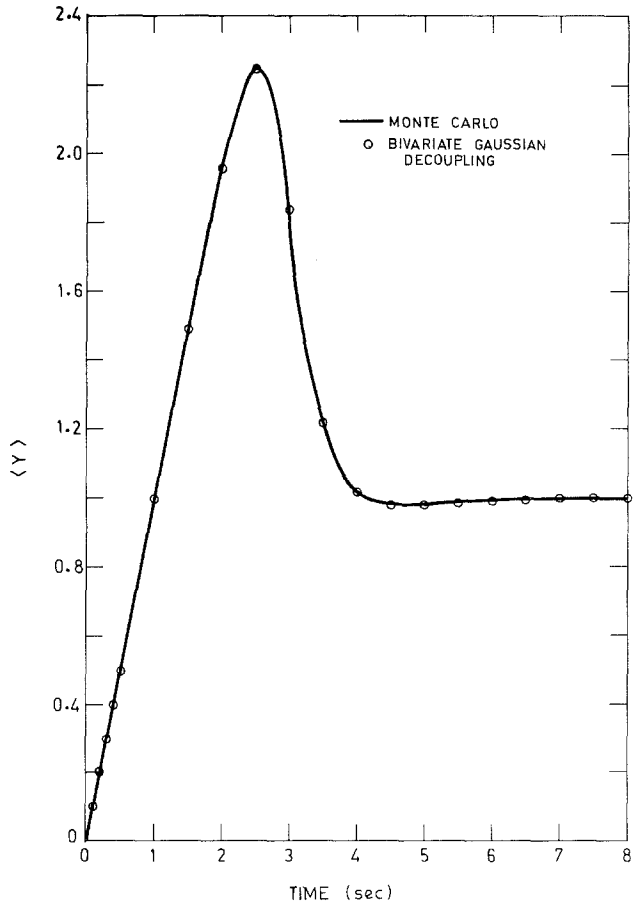


Fig. 5. $\langle y\rangle$ as a function of time for the bivariate unstable case where only one stable steady state is allowed. The values of the parameters used as $\epsilon_1 = \epsilon_2 = 5 \times 10^{-7}$ and deterministic initial conditions $\langle x(0)\rangle = x(0) = 0.03$ and $\langle y(0)\rangle = y(0) = 0$.

and

$$\langle x^3 y\rangle = 3\langle x^2\rangle\langle xy\rangle - 2\langle x\rangle^3\langle y\rangle \tag{33f}$$

These now can be used in Eqs. (24) to (28) to solve the resulting five coupled equations. The results of $\langle y\rangle$ and $\sigma_2^2$ have been shown in Figs. 5 and 6 respectively, along with the Monte Carlo results. The results reported are for $b_0 = 1$, $b_1 = 2$, $b_2 = 1$, $\epsilon_1 = \epsilon_2 = 5 \times 10^{-7}$, $\langle x(0)\rangle = 0.03$, $\langle x^2(0)\rangle = 0.0009$, $\langle y(0)\rangle = 0$, $\langle y^2(0)\rangle = 0$, and $\langle x(0)y(0)\rangle = 0$. The value of $\epsilon_1$ has been chosen to be of the order of inverse size of the system. (The maximum dislocation density is $\sim 10^{14}$.) From these figures, it is clear that the decoupling scheme gives results on the first and second cumulants which are in excellent agreement with the Monte Carlo results. Results on $\langle x\rangle$, $\langle x^2\rangle$, and $\langle xy\rangle$ also agree very well. Again the expected fluctuation

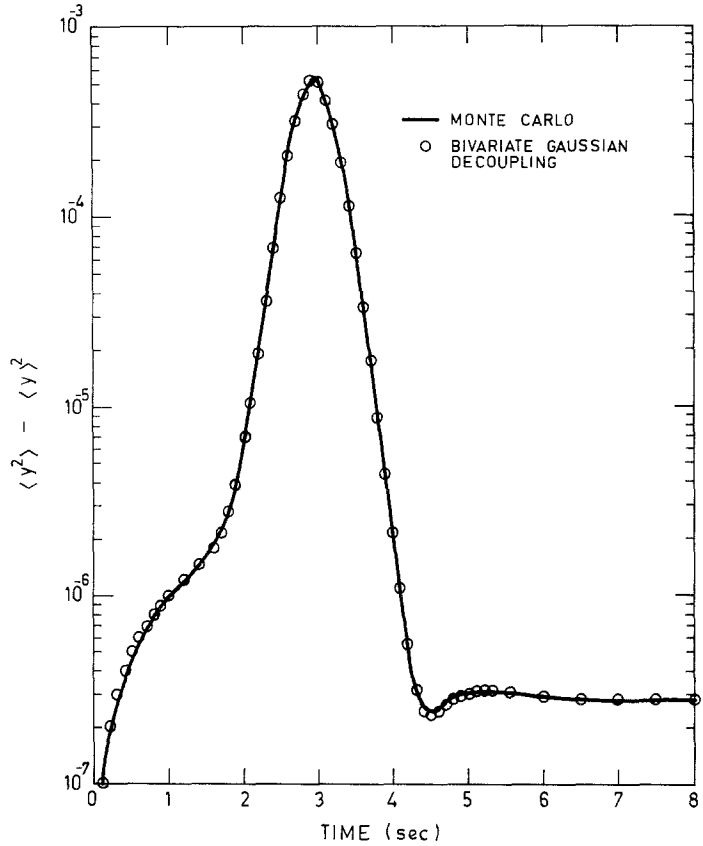


Fig. 6. The variance of $y$ for the problem considered in Fig. 5.

enhancement is predicted correctly. In the Appendix we have shown that the POL procedure gives identical equations. From the Appendix it should be clear that the procedure via POL becomes more and more cumbersome in higher dimensions.

## 4. DISCUSSION

We have shown that a logical extension of the POL procedure leads to the Gaussian decoupling scheme. The method is simple and yields good results both in the stable and the unstable cases. The method also gives some insight into the unstable case when multiple steady states are possible. In such cases we see that if we respect the symmetry of the distribution function, it should give a reasonable decoupling scheme. We have shown that the method can be easily generalized to the bivariate case. The usual fluctuation enhancement that is expected during a passage from an unstable state to a stable state is also properly reproduced. The agreement with Monte Carlo technique demonstrates that the decoupling scheme should be of great value in higher-dimensional problems (where the Monte Carlo method becomes prohibitive), particularly when the interest can be restricted to the first two moments.

## NOTE ADDED IN PROOF

It should be noted that the bimodal Gaussian decoupling has been used earlier by Langer et al.[13] in the context of spinodal decomposition.

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## APPENDIX

Consider writing the equations (21) and (22) in the linearized forms

$$\frac{dx}{dt} = \alpha_1 x + \beta_1 y + C_1 + \eta_1 \tag{A.1}$$

and

$$\frac{dy}{dt} = \alpha_2 x + \beta_2 y + C_2 + \eta_2 \tag{A.2}$$

where the errors in the above linearization are given by

$$e_1 = xy - x^2 - \alpha_1 x - \beta_1 y - C_1 \tag{A.3}$$

and

$$e_2 = b_0 - b_1 x^2 y + b_2 x^3 - \alpha_2 x - \beta_2 y - C_2 \tag{A.4}$$

Minimizing the average of the square of these errors, we get

$$C_1 = \langle xy \rangle - \langle x^2 \rangle - \alpha_1 \langle x \rangle - \beta_1 \langle y \rangle \tag{A.5}$$

and

$$C_2 = b_0 - b_1 \langle x^2 y \rangle + b_2 \langle x^3 \rangle - \alpha_2 \langle x \rangle - \beta_2 \langle y \rangle \tag{A.6}$$

with

$$\begin{pmatrix} \alpha_1 \\ \beta_2 \end{pmatrix} = \frac{1}{\sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2)} \begin{pmatrix} \sigma_2^2 R_1 - \sigma_1 \sigma_2 \rho_{12} R_2 \\ \sigma_1^2 R_2 - \sigma_1 \sigma_2 \rho_{12} R_1 \end{pmatrix} \tag{A.7}$$

and

$$\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \frac{1}{\sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2)} \begin{pmatrix} \sigma_2^2 R_3 - \sigma_1 \sigma_2 \rho_{12} R_4 \\ \sigma_1^2 R_4 - \sigma_1 \sigma_2 \rho_{12} R_3 \end{pmatrix} \tag{A.8}$$

In the above expressions

$$R_1 = \langle x^2 y \rangle - \langle x^3 \rangle - \langle x \rangle \langle xy \rangle + \langle x \rangle \langle x^2 \rangle \tag{A.9}$$

$$R_2 = \langle xy^2 \rangle - \langle x^2 y \rangle + \langle y \rangle \langle x^2 \rangle - \langle y \rangle \langle xy \rangle \tag{A.10}$$

$$R_3 = b_1 \left[ \langle x \rangle \langle x^2 y \rangle - \langle x^3 y \rangle \right] + b_2 \left[ \langle x^4 \rangle - \langle x \rangle \langle x^3 \rangle \right] \tag{A.11}$$

and

$$R_4 = b_1 \left[ \langle x^2 y \rangle \langle y \rangle - \langle x^2 y^2 \rangle \right] + b_2 \left[ \langle x^3 y \rangle - \langle x^3 \rangle \langle y \rangle \right] \tag{A.12}$$

Finally, since the above linearized equations imply a Gaussian process, the moments $\langle x^3 \rangle$, $\langle x^4 \rangle$, $\langle x^2 y \rangle$, $\langle xy^2 \rangle$, $\langle x^2 y^2 \rangle$, and $\langle x^3 y \rangle$ can be expressed in terms of the first two moments of each of the variables and the correlation coefficient. It is straightforward to check that when this is done, they yield rate equations identical to those we have used for the first two moments.

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